Symmetry Groups, Orbifolds and Tilings

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Enumerating Tilings

Theorem *Given a 2d symmetry group and* k *2* N. *All tile-*k*-transitive periodic tilings with group can be* recursively *enumerated using the algorithms FUNDAMENTAL, SPLIT and GLUE.*

SPLIT: Delone, Dolibilin & Stogrin[78]

GLUE: L. Zamorzaeva[84]

For Delaney symbols - Huson[93]

The Form of an Orbifold

Two orbifold symbols O and O' are *of the same form*, if there exists a permutation h of $\mathbb{N} \setminus \{0, 1\}$, such that $h_*(O) = O'$.

Example: If h exchanges 3 and 4, then

$$h_*(\check{Z}\check{Z}245*633) = \check{Z}\check{Z}235*644.$$

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Main Result

For an orbifold O, let $\mathbf{T}(O)$ denote the set of all periodic tilings whose symmetry group correspond to O.

Theorem *If O and O^0 are of the same form, then there exists a one–one-correpondence between $\mathbf{T}(O)$ and $\mathbf{T}(O^0)$, unless one of the orbifolds is $p22$, Ł$p22$, or 2 Ł p, with p ½ 2.*

(Balke & H., to appear in: Geom. Ded.)

Given the form of an orbifold, all corresponding periodic tilings can be systematically enumerated.

Ribbon Symmetry Groups

Set n = *1*:

Ł22n	Ł22 *1*	pmm2
2Łn	2Ł *1*	pma2
22n	22 *1*	p112
Łnn	Ł *1* *1*	pm11
nŁ	*1* Ł	p1m1
nð	*1* ð	p1a1
nn	*1* *1*	p111

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Classifying Ribbon Tilings

Theorem *The classification of tile-k-transitive tilings of the ribbon* **R** *(or pinched-ribbon* **pR***) can be derived from the classification of all longitudal tile-k-transitive tilings of the sphere.*

Form	Orbifold	Group	Tile-trans.		Tile-2-trans.		Tile-3-t
			R	**pR**	**R**	**pR**	**R**
2Łn	2Ł 1	pma2	7	2	119	22	2326
Ł22n	Ł22 1	pmm2	4	2	90	42	2189
Łnn	Ł 1 1	pm11	2	1	23	10	406
22n	22 1	p112	5	1	73	14	1150
nŁ	1 Ł	p1m1	2	0	23	3	271
nn	1 1	p111	1	0	7	1	57
nð	1 ð	p1a1	3	0	23	1	251
		Total	24	6	358	93	6650

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Identifying the Crystallographic Group from a Triangulated 3D Orbifold

Examples:

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Orbifolds

A closed diÞ. n-*orbifold* Q is a HausdorÞ space X^n, together with (compatible) modellings of neighborhoods of each point in X^n on $\mathbf{R}^n/$(þnite subgroup of $O(n)$), where the point corresponds to the equivalence class of the origin. The *singular set* S_Q of Q consists of those points for which the þnite group is trivial (Dunbar 88).

Let X be a simply-connected metric space and G Isom(X), with compact fundamental domain. Then the quotient G *n* X gives rise to an orbifold.

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3D Euclidean Space Groups

There are 230 types of 3D crystallographic space groups, i.e. 219 isomorphism types, and 11 pairs that only diÞer in their left- or right-handedness (Federov 1890?).

Problem *Given the triangulation of a 3D orbifold Q, in terms of a Delaney symbol (D, m), assumed to be euclidean.*
How to determine the crystallographic type of the associated group?

The Orbifold Graph

Given the triangulation of a 3D orbifold Q in terms of a Delaney symbol (D, m). The *orbifold graph* describes the singular set S_Q of Q combinatorially, i.e. without the embedding. It can be easily computed from (D, m).

Examples:

Results

Given a list of Delaney symbols, one of each of the 219 types of groups, we obtain the following results:

ž The 219 types of 3D crystallographic groups give rise to 189 diÞerent orbifold graphs (taking orientability into account)

ž 175 groups can be identiþed solely by their orbifold graph

ž The remaining 44 groups can all be destinguished using \abelian invariants" (O. Delgado)